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Fuzzy subsemigroups and fuzzy ideals with operators in semigroups

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Abstract. Given a set Ω, the notion of an Ω-fuzzy subsemigroup in semigroups is given. Using fuzzy subsemigroups, an Ω-fuzzy subsemigroup is described. Conversely, a fuzzy subsemigroup is constructed by using an Ω-fuzzy subsemigroup. How the homomorphic images and inverse images of Ω-fuzzy subsemigroups become Ω-fuzzy subsemigroups is stated. The notion of an Ω-fuzzy left (right) ideal (an Ω-fuzzy bi-ideal, an Ω-fuzzy interior ideal) generated by an Ω-fuzzy set in semigroups is introduced. The depictions of them are examined.

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1. Introduction

Hong et al. [1] and Kuroki [2, 3] have studied several properties of fuzzy left (right) ideals, fuzzy bi-ideals and fuzzy interior ideals in semigroups. For more other study on the fuzzy theory in semigroups, we refer to papers [4, 7, 9, 10]. In this paper, by using a set Ω, we define Ω-fuzzy subsemigroups, Ω-fuzzy left (right) ideals, Ω-fuzzy bi-ideals and Ω-fuzzy interior ideals in semigroups. We describe an Ω-fuzzy subsemigroup by using a fuzzy subsemigroup and vice versa. We state how the homomorphic images and inverse images of Ω-fuzzy subsemigroups become Ω-fuzzy subsemigroups. We deal with the notion of an Ω-fuzzy left (right) ideal (an Ω-fuzzy bi-ideal, an Ω-fuzzy interior ideal) generated by an Ω-fuzzy set in semigroups, and examine the depictions of them.

2. Preliminaries

A semigroup S is said to be *regular* if it satisfies:

$$(\forall a \in S)\,(\exists x \in S)\,(a = axa).$$

A semigroup S is said to be *completely regular* if it satisfies:

$$(\forall a \in S)\,(\exists x \in S)\,(a = axa,\ ax = xa).$$

By a *subsemigroup* of a semigroup S we mean a nonempty subset A of S such that $A^2 \subseteq A$, and by a *left* (*right*) *ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a nonempty subset of a semigroup S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. A nonempty subset A of a semigroup S is called a *generalized bi-ideal* of S if $ASA \subseteq A$ (see Lajos [6]).

A *fuzzy set* in S is a function μ from S into the unit interval $[0, 1]$. A fuzzy set μ in S is called a *fuzzy subsemigroup* of S if it satisfies

$$(\forall x, y \in S)\,(\mu(xy) \geq \min\{\mu(x), \mu(y)\}),$$

and is called a *fuzzy left* (*right*) *ideal* of S if

$$(\forall x, y \in S)\,(\mu(xy) \geq \mu(y) \quad (\mu(xy) \geq \mu(x))).$$

If μ is both a fuzzy left and a fuzzy right ideal of S, we say that μ is a *fuzzy ideal* of S. A fuzzy subsemigroup μ of S is called a *fuzzy bi-ideal* of S if it satisfies

$$(\forall w, x, y \in S)\,(\mu(xwy) \geq \min\{\mu(x), \mu(y)\}).$$

3. Ω-fuzzy subsemigroups

In what follows let S and Ω denote a semigroup and a nonempty set, respectively, unless otherwise specified. A mapping $f : S \times \Omega \to [0, 1]$ is called an Ω-*fuzzy set* in S.

Definition 3.1. An Ω-fuzzy set f in S is called an Ω-*fuzzy subsemigroup* of S if it satisfies

$$(\forall \alpha \in \Omega)\,(\forall x, y \in S)\,(f(xy, \alpha) \geq \min\{f(x, \alpha), f(y, \alpha)\}).$$

Example 3.2. Consider a semigroup $S = \{a, b\}$ with the following Cayley table:

	a	b
a	a	b
b	b	a

Let $\Omega = \{1, 2\}$ and let f be an Ω-fuzzy set in S defined by $f(a, 1) = f(a, 2) = 1$, $f(b, 1) = 0.8$ and $f(b, 2) = 0.5$. It is easy to verify that f is an Ω-fuzzy subsemigroup of S.

Example 3.3. Let $S^\Omega := \{u \mid u : \Omega \to S\}$. For any $u, v \in S^\Omega$, we define $(uv)(\alpha) = u(\alpha)v(\alpha)$ for all $\alpha \in \Omega$. Then S^Ω is a semigroup. Let μ be a fuzzy subsemigroup of S and let $\Phi : S^\Omega \times \Omega \to [0, 1]$ be a function defined by $\Phi(u, \alpha) = \mu(u(\alpha))$ for all $u \in S^\Omega$ and $\alpha \in \Omega$. Then Φ is an Ω-fuzzy subsemigroup of S^Ω.

Theorem 3.4. *Let A be a subsemigroup of S^Ω. Then for any $\beta \in \Omega$, the set*

$$A_\beta := \{u(\beta) \mid u \in A\}$$

is a subsemigroup of S.

Proof. For any $\beta \in \Omega$, let $u(\beta), v(\beta) \in A_\beta$. Then $u(\beta)v(\beta) = (uv)(\beta) \in A_\beta$ since $uv \in A$. Hence A_β is a subsemigroup of S. □

Proposition 3.5. *If f is an Ω-fuzzy subsemigroup of S, then a fuzzy set $f_\alpha : S \to [0,1]$, $\alpha \in \Omega$, given by $f_\alpha(x) = f(x, \alpha)$ for all $x \in S$ is a fuzzy subsemigroup of S.*

Proof. Let $x, y \in S$. Then

$$f_\alpha(xy) = f(xy, \alpha) \geq \min\{f(x,\alpha), f(y,\alpha)\} = \min\{f_\alpha(x), f_\alpha(y)\}.$$

This completes the proof. □

Proposition 3.6. *If $f_\alpha, \alpha \in \Omega$, is a fuzzy subsemigroup of S, then a function $f : S \times \Omega \to [0,1]$, $(x, \alpha) \mapsto f_\alpha(x)$, is an Ω-fuzzy subsemigroup of S.*

Proof. For any $x, y \in S$, we have

$$f(xy, \alpha) = f_\alpha(xy) \geq \min\{f_\alpha(x), f_\alpha(y)\} = \min\{f(x,\alpha), f(y,\alpha)\}.$$

Hence f is an Ω-fuzzy subsemigroup of S. □

Theorem 3.7. *Let Φ be a fuzzy subsemigroup of S^Ω and let f be an Ω-fuzzy set in S defined by*

$$f(x, \alpha) := \sup\{\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = x\}$$

for all $x \in S$ and $\alpha \in \Omega$. Then f is an Ω-fuzzy subsemigroup of S.

Proof. Let $x, y \in S$ and $\alpha \in \Omega$. Then

$$\begin{aligned} f(xy,\alpha) &= \sup\{\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = xy\} \\ &\geq \sup\{\Phi(uv) \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\} \\ &\geq \sup\{\min\{\Phi(u), \Phi(v) \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\} \\ &= \min\{\sup\{\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = x\}, \sup\{\Phi(v) \mid v \in S^\Omega,\ v(\alpha) = y\}\} \\ &= \min\{f(x,\alpha), f(y,\alpha)\}. \end{aligned}$$

Hence f is an Ω-fuzzy subsemigroup of S. □

Example 3.8. Let $S = \{a, b\}$ be a semigroup in Example 3.2 and let $\Omega := \{1, 2\}$. Then $S^\Omega := \{e, u, v, w\}$, where $e(1) = e(2) = v(1) = w(2) = a$ and $u(1) = u(2) = v(2) = w(1) = b$, is a semigroup (in fact, a commutative group) under the following Cayley table:

	e	u	v	w
e	e	u	v	w
u	u	e	w	v
v	v	w	e	u
w	w	v	u	e

Let Φ be a fuzzy set in S^Ω defined by $\Phi(e) = 0.9$, $\Phi(u) = \Phi(v) = 0.2$, and $\Phi(w) = 0.7$. Then Φ is a fuzzy subsemigroup of S^Ω. Thus we can obtain an Ω-fuzzy subsemigroup f of S as follows:

$$f(a, 1) = \sup\{\Phi(\heartsuit) \mid \heartsuit \in S^\Omega,\ \heartsuit(1) = a\} = \sup\{\Phi(e), \Phi(v)\} = 0.9,$$

$$f(a,2) = \sup\{\Phi(\heartsuit) \mid \heartsuit \in S^\Omega,\ \heartsuit(2) = a\} = \sup\{\Phi(e), \Phi(w)\} = 0.9,$$
$$f(b,1) = \sup\{\Phi(\heartsuit) \mid \heartsuit \in S^\Omega,\ \heartsuit(1) = b\} = \sup\{\Phi(u), \Phi(w)\} = 0.7,$$
$$f(b,2) = \sup\{\Phi(\heartsuit) \mid \heartsuit \in S^\Omega,\ \heartsuit(2) = b\} = \sup\{\Phi(u), \Phi(v)\} = 0.2.$$

Theorem 3.9. *Let f be an Ω-fuzzy subsemigroup of S and let Φ be a fuzzy set in S^Ω defined by*

$$\Phi(u) = \inf\{f(u(\alpha), \alpha) \mid \alpha \in \Omega\}$$

for all $u \in S^\Omega$. Then Φ is a fuzzy subsemigroup of S^Ω.

Proof. For any $u, v \in S^\Omega$, we have

$$\begin{aligned} \Phi(uv) &= \inf\{f((uv)(\alpha), \alpha) \mid \alpha \in \Omega\} \\ &= \inf\{f(u(\alpha)v(\alpha), \alpha) \mid \alpha \in \Omega\} \\ &\geq \inf\{\min\{f(u(\alpha), \alpha), f(v(\alpha), \alpha) \mid \alpha \in \Omega\} \\ &= \min\{\inf\{f(u(\alpha), \alpha) \mid \alpha \in \Omega\}, \inf\{f(v(\alpha), \alpha) \mid \alpha \in \Omega\}\} \\ &= \min\{\Phi(u), \Phi(v)\}. \end{aligned}$$

Thus Φ is a fuzzy subsemigroup of S^Ω. □

Example 3.10. Let f be the Ω-fuzzy subsemigroup of S in Example 3.2 and let S^Ω be the commutative group in Example 3.8. Then we can induce a fuzzy subsemigroup of S^Ω as follows:

$$\begin{aligned} \Phi(e) &= \inf\{f(e(\alpha), \alpha) \mid \alpha \in \Omega\} = \inf\{f(e(1), 1), f(e(2), 2)\} \\ &= \inf\{f(a,1), f(a,2)\} = 1, \end{aligned}$$
$$\begin{aligned} \Phi(u) &= \inf\{f(u(\alpha), \alpha) \mid \alpha \in \Omega\} = \inf\{f(u(1), 1), f(u(2), 2)\} \\ &= \inf\{f(b,1), f(b,2)\} = 0.5, \end{aligned}$$
$$\begin{aligned} \Phi(v) &= \inf\{f(v(\alpha), \alpha) \mid \alpha \in \Omega\} = \inf\{f(v(1), 1), f(v(2), 2)\} \\ &= \inf\{f(a,1), f(b,2)\} = 0.5, \end{aligned}$$
$$\begin{aligned} \Phi(w) &= \inf\{f(w(\alpha), \alpha) \mid \alpha \in \Omega\} = \inf\{f(w(1), 1), f(w(2), 2)\} \\ &= \inf\{f(b,1), f(a,2)\} = 0.8. \end{aligned}$$

Definition 3.11. Let $\varphi : S \to T$ be a homomorphism of semigroups and let g be an Ω-fuzzy set in T. Then the *inverse image* of g, denoted by $\varphi^{-1}[g]$, is the Ω-fuzzy set in S given by $\varphi^{-1}[g](x, \alpha) = g(\varphi(x), \alpha)$ for all $x \in S$ and $\alpha \in \Omega$. Conversely, let f be an Ω-fuzzy set in S. The *image* of f, written as $\varphi[f]$, is an Ω-fuzzy set in T defined by

$$\varphi[f](y, \alpha) = \begin{cases} \sup\limits_{z \in \varphi^{-1}(y)} f(z, \alpha) & \text{if } \varphi^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases}$$

for all $y \in T$ and $\alpha \in \Omega$, where $\varphi^{-1}(y) = \{x \mid \varphi(x) = y\}$.

Theorem 3.12. *Let $\varphi : S \to T$ be a homomorphism of semigroups. If g is an Ω-fuzzy subsemigroup of T, then the inverse image $\varphi^{-1}[g]$ of g is an Ω-fuzzy subsemigroup of S.*

Proof. Let $x, y \in S$ and $\alpha \in \Omega$. Then

$$\begin{aligned} \varphi^{-1}[g](xy, \alpha) &= g(\varphi(xy), \alpha) = g(\varphi(x)\varphi(y), \alpha) \\ &\geq \min\{g(\varphi(x), \alpha), g(\varphi(y), \alpha)\} = \min\{\varphi^{-1}[g](x, \alpha), \varphi^{-1}[g](y, \alpha)\}. \end{aligned}$$

Hence $\varphi^{-1}[g]$ is an Ω-fuzzy subsemigroup of S. □

Theorem 3.13. *Let $\varphi : S \to T$ be a homomorphism between semigroups S and T. If f is an Ω-fuzzy subsemigroup of S, then the image $\varphi[f]$ of f is an Ω-fuzzy subsemigroup of T.*

Proof. We first prove that

$$\varphi^{-1}(y_1)\varphi^{-1}(y_2) \subseteq \varphi^{-1}(y_1y_2) \tag{3.1}$$

for all $y_1, y_2 \in T$. For, if $x \in \varphi^{-1}(y_1)\varphi^{-1}(y_2)$, then $x = x_1x_2$ for some $x_1 \in \varphi^{-1}(y_1)$ and $x_2 \in \varphi^{-1}(y_2)$. Since f is a homomorphism, it follows that $\varphi(x) = \varphi(x_1x_2) = \varphi(x_1)\varphi(x_2) = y_1y_2$ so that $x \in \varphi^{-1}(y_1y_2)$. Hence (3.1) holds. Now let $y_1, y_2 \in T$ and $\alpha \in \Omega$. Assume that $y_1y_2 \notin \mathrm{Im}(\varphi)$. Then $\varphi[f](y_1y_2, \alpha) = 0$. But if $y_1y_2 \notin \mathrm{Im}(\varphi)$, i.e., $\varphi^{-1}(y_1y_2) = \varnothing$, then $\varphi^{-1}(y_1) = \varnothing$ or $\varphi^{-1}(y_2) = \varnothing$ by (3.1). Thus $\varphi[f](y_1, \alpha) = 0$ or $\varphi[f](y_2, \alpha) = 0$, and so

$$\varphi[f](y_1y_2, \alpha) = 0 = \min\{\varphi[f](y_1, \alpha), \varphi[f](y_2, \alpha)\}.$$

Suppose that $\varphi^{-1}(y_1y_2) \neq \varnothing$. Then we should consider two cases as follows:

$$\text{(i)}\quad \varphi^{-1}(y_1) = \varnothing \quad\text{or}\quad \varphi^{-1}(y_2) = \varnothing,$$
$$\text{(ii)}\quad \varphi^{-1}(y_1) \neq \varnothing \quad\text{and}\quad \varphi^{-1}(y_2) \neq \varnothing.$$

For the case (i), we have $\varphi[f](y_1, \alpha) = 0$ or $\varphi[f](y_2, \alpha) = 0$, and so

$$\varphi[f](y_1y_2, \alpha) \geq 0 = \min\{\varphi[f](y_1, \alpha), \varphi[f](y_2, \alpha)\}.$$

Case (ii) implies from (3.1) that

$$\begin{aligned}
&\varphi[f](y_1y_2, \alpha) = \sup_{z \in \varphi^{-1}(y_1y_2)} f(z, \alpha) \geq \sup_{z \in \varphi^{-1}(y_1)\varphi^{-1}(y_2)} f(z, \alpha) \\
&= \sup_{x_1 \in \varphi^{-1}(y_1),\, x_2 \in \varphi^{-1}(y_2)} f(x_1x_2, \alpha) \\
&\geq \sup_{x_1 \in \varphi^{-1}(y_1),\, x_2 \in \varphi^{-1}(y_2)} \min\{f(x_1, \alpha), f(x_2, \alpha)\} \\
&= \min\{\sup_{x_1 \in \varphi^{-1}(y_1)} f(x_1, \alpha), \sup_{x_2 \in \varphi^{-1}(y_2)} f(x_2, \alpha)\} \\
&= \min\{\varphi[f](y_1, \alpha), \varphi[f](y_2, \alpha)\}.
\end{aligned}$$

Hence $\varphi[f](y_1y_2, \alpha) \geq \min\{\varphi[f](y_1, \alpha), \varphi[f](y_2, \alpha)\}$ for all $y_1, y_2 \in T$ and $\alpha \in \Omega$. This completes the proof. □

4. Ω-fuzzy bi-ideals and Ω-fuzzy interior ideals

Definition 4.1. An Ω-fuzzy set f in S is called an *Ω-fuzzy left* (resp. *right*) *ideal* of S if

$$(\forall x, y \in S)\,(\forall \alpha \in \Omega)\,(f(xy, \alpha) \geq f(y, \alpha) \text{ (resp. } f(xy, \alpha) \geq f(x, \alpha))).$$

If f is both an Ω-fuzzy left and an Ω-fuzzy right ideal of S, we say that f is an *Ω-fuzzy ideal* of S.

Let f be an Ω-fuzzy left (right) ideal of S and let Φ be the fuzzy set in S^Ω given in Theorem 3.9. For any $u, v \in S^\Omega$, we have

$$\begin{aligned}
\Phi(uv) &= \inf\{f((uv)(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \inf\{f(u(\alpha)v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&\geq \inf\{f(v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \Phi(v).
\end{aligned}$$

Similarly $\Phi(uv) \geq \Phi(u)$. Hence Φ is a fuzzy ideal of S^Ω.

Definition 4.2. An Ω-fuzzy set f in S is called an Ω-*fuzzy bi-ideal* of S if it satisfies

(i) f is an Ω-fuzzy subsemigroup of S,

(ii) $(\forall \alpha \in \Omega)\,(\forall x, y, z \in S)\,(f(xyz, \alpha) \geq \min\{f(x, \alpha), f(z, \alpha)\})$.

If f satisfies the condition (ii) only, we say that f is a *generalized* Ω-*fuzzy bi-ideal* of S. It is clear that every Ω-fuzzy bi-ideal of S is a generalized Ω-fuzzy bi-ideal of S, but not conversely as seen in the following example.

Example 4.3. (1) Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\phi : S \times \Omega \to [0, 1]$ be a fuzzy set in $S \times \Omega$ defined by $f(a, \alpha) = 0.5$, $f(b, \alpha) = f(d, \alpha) = 0$, and $f(c, \alpha) = 0.2$. Then f is a generalized Ω-fuzzy bi-ideal of S, which is not an Ω-fuzzy bi-ideal of S since

$$f(cc, \alpha) = f(b, \alpha) = 0 \ngeq 0.2 = \min\{f(c, \alpha), f(c, \alpha)\}.$$

(2) In Example 3.3, if μ is a fuzzy bi-ideal of S, then Φ is an Ω-fuzzy bi-ideal of S^Ω.

(3) The Ω-fuzzy subsemigroup f in Example 3.2 is not an Ω-fuzzy bi-ideal of S since

$$f(aba, 1) = f(b, 1) = 0.8 < 1 = f(a, 1) = \min\{f(a, 1), f(a, 1)\}.$$

Example 4.4. Let $\Omega := \{\mu \mid \mu \text{ is a fuzzy bi-ideal of } S\}$ and let f be a mapping from $S \times \Omega$ into $[0, 1]$ defined by $f(x, \mu) = \mu(x)$ for all $x \in S$ and $\mu \in \Omega$. Then f is an Ω-fuzzy bi-ideal of S.

Proposition 4.5. *If $f_\alpha, \alpha \in \Omega$, is a fuzzy bi-ideal of S, then a function $\phi : S \times \Omega \to [0, 1]$, $(x, \alpha) \mapsto f_\alpha(x)$, is an Ω-fuzzy bi-ideal of S.*

Proof. According to Proposition 3.6, f is an Ω-fuzzy subsemigroup of S. Let $x, y, z \in S$ and $\alpha \in \Omega$. Then

$$f(xyz, \alpha) = f_\alpha(xyz) \geq \min\{f_\alpha(x), f_\alpha(z)\} = \min\{f(x, \alpha), f(z, \alpha)\}.$$

This completes the proof. □

Proposition 4.6. *Let f be an Ω-fuzzy bi-ideal of S and $\alpha \in \Omega$. Define a function $f_\alpha : S \to [0, 1]$ by $f_\alpha(x) = f(x, \alpha)$ for all $x \in S$. Then f_α is a fuzzy bi-ideal of S.*

Proof. According to Proposition 3.5, f_α is a fuzzy subsemigroup of S. For any $x, y, z \in S$ we have

$$f_\alpha(xyz) = f(xyz, \alpha) \geq \min\{f(x, \alpha), f(y, \alpha)\} = \min\{f_\alpha(x), f_\alpha(z)\},$$

and so f_α is a fuzzy bi-ideal of S. □

Theorem 4.7. *If S is a group, then every Ω-fuzzy bi-ideal of S is a constant function.*

Proof. Let f be an Ω-fuzzy bi-ideal of S. For any $x \in S$ and $\alpha \in \Omega$, we get

$$\begin{aligned} f(x,\alpha) &= f(exe,\alpha) \geq \min\{f(e,\alpha), f(e,\alpha)\} = f(e,\alpha) \\ &= f(ee,\alpha) = f((xx^{-1})(x^{-1}x),\alpha) = f(x(x^{-1}x^{-1})x),\alpha) \\ &\geq \min\{f(x,\alpha), f(x,\alpha)\} = f(x,\alpha), \end{aligned}$$

where e is the identity element of S. Hence $f(x,\alpha) = f(e,\alpha)$, and f is a constant function. □

Theorem 4.8. *If A is a bi-ideal of S, then the characteristic function $\chi_{A\times\Omega}$ of $A \times \Omega$ is an Ω-fuzzy bi-ideal of S.*

Proof. Let $\alpha \in \Omega$ and let $x, y, z \in S$. If $x \notin A$ or $y \notin A$, then $(x,\alpha) \notin A \times \Omega$ or $(y,\alpha) \notin A \times \Omega$. Hence

$$\chi_{A\times\Omega}(xy,\alpha) \geq 0 = \min\{\chi_{A\times\Omega}(x,\alpha), \chi_{A\times\Omega}(y,\alpha)\}.$$

If $x \in A$ and $y \in A$, then $xy \in A$ and so

$$\chi_{A\times\Omega}(xy,\alpha) = \min\{\chi_{A\times\Omega}(x,\alpha), \chi_{A\times\Omega}(y,\alpha)\}.$$

Now if $x \in A$ and $z \in A$, then we have $\chi_{A\times\Omega}(x,\alpha) = \chi_{A\times\Omega}(z,\alpha) = 1$. Since A is a bi-ideal of S, it follows that $(xyz,\alpha) \in ASA \times \Omega \subseteq A \times \Omega$ so that

$$\chi_{A\times\Omega}(xyz,\alpha) = 1 = \min\{\chi_{A\times\Omega}(x,\alpha), \chi_{A\times\Omega}(z,\alpha)\}.$$

If $x \notin A$ or $z \notin A$, then $\chi_{A\times\Omega}(x,\alpha) = 0$ or $\chi_{A\times\Omega}(z,\alpha) = 0$. Thus

$$\chi_{A\times\Omega}(xyz,\alpha) \geq 0 = \min\{\chi_{A\times\Omega}(x,\alpha), \chi_{A\times\Omega}(z,\alpha)\}.$$

Therefore $\chi_{A\times\Omega}$ is an Ω-fuzzy bi-ideal of S. □

Theorem 4.9. *Let A be a subset of S such that the characteristic function $\chi_{A\times\Omega}$ of $A \times \Omega$ is a generalized Ω-fuzzy bi-ideal of S. Then A is a generalized bi-ideal of S.*

Proof. Let $\alpha \in \Omega$, $x, z \in A$ and $y \in S$. Then $\chi_{A\times\Omega}(x,\alpha) = 1 = \chi_{A\times\Omega}(z,\alpha)$. Since $\chi_{A\times\Omega}$ is a generalized Ω-fuzzy bi-ideal of S, we have

$$\chi_{A\times\Omega}(xyz,\alpha) \geq \min\{\chi_{A\times\Omega}(x,\alpha), \chi_{A\times\Omega}(z,\alpha)\} = 1$$

and so $\chi_{A\times\Omega}(xyz,\alpha) = 1$. Thus $(xyz,\alpha) \in A \times \Omega$ and hence $xyz \in A$. This shows that A is a generalized bi-ideal of S. □

We give a condition for a generalized Ω-fuzzy bi-ideal to be an Ω-fuzzy bi-ideal.

Theorem 4.10. *Every generalized Ω-fuzzy bi-ideal of a regular semigroup S is an Ω-fuzzy bi-ideal of S.*

Proof. Let f be a generalized Ω-fuzzy bi-ideal of a regular semigroup S and let $\alpha \in \Omega$ and $x, y \in S$. Since S is regular, it follows that there exists $a \in S$ such that $y = yay$ so that

$$f(xy,\alpha) = f(x(yay),\alpha) = f(x(ya)y,\alpha) \geq \min\{f(x,\alpha), f(y,\alpha)\}.$$

This means that f is an Ω-fuzzy subsemigroup of S, and so it is an Ω-fuzzy bi-ideal of S. □

Proposition 4.11. *Let f be an Ω-fuzzy bi-ideal of S. If S is completely regular, then*

$$(\forall x \in S)\,(\forall \alpha \in \Omega)\,(f(x,\alpha) = f(x^2,\alpha)).$$

Proof. Let $\alpha \in \Omega$ and let x be any element of S. Then there exists $a \in S$ such that $x = x^2ax^2$, which implies that

$$\begin{aligned} f(x,\alpha) = f(x^2ax^2,\alpha) &\geq \min\{f(x^2,\alpha), f(x^2,\alpha)\} \\ = f(x^2,\alpha) &\geq \min\{f(x,\alpha), f(x,\alpha)\} = f(x,\alpha) \end{aligned}$$

so that $f(x,\alpha) = f(x^2,\alpha)$. This completes the proof. □

Lemma 4.12. [8, p.105] *For a semigroup S the following conditions are equivalent:*

(i) *S is completely regular.*
(ii) *S is a union of groups.*
(iii) *$a \in a^2Sa^2$ for all $a \in S$.*

Lemma 4.13. [5, Theorem 1] *A semigroup S is a semilattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets.*

Proposition 4.14. *Let f be an Ω-fuzzy bi-ideal of S. If S is a semilattice of groups, then*

$$(\forall x, y \in S)\,(\forall \alpha \in \Omega)\,(f(x,\alpha) = f(x^2,\alpha),\; f(xy,\alpha) = f(yx,\alpha)).$$

Proof. If S is a semilattice of groups, then S is a union of groups and so S is completely regular by Lemma 4.12. Hence $f(x,\alpha) = f(x^2,\alpha)$ for all $x \in S$ and $\alpha \in \Omega$ by Proposition 4.11. Using Lemma 4.13, we have

$$\begin{aligned} (xy)^3 = (xyx)(yxy) &\in B[xyx]B[yxy] = B[yxy](B[xyx])^2 \\ &\subseteq B[yxy]SB[xyx] \subseteq yxySxyx \subseteq yxSyx \end{aligned}$$

for all $x, y \in S$, where $B[a]$ means the principal bi-ideal of S generated by $a \in S$. This implies that there exists $a \in S$ such that $(xy)^3 = (yx)a(yx)$. Thus for any $\alpha \in \Omega$, we get

$$\begin{aligned} f(xy,\alpha) &= f((xy)^3,\alpha) = f((yx)a(yx),\alpha) \\ &\geq \min\{f(yx,\alpha), f(yx,\alpha)\} = f(yx,\alpha). \end{aligned}$$

Similarly, we obtain $f(yx,\alpha) \geq f(xy,\alpha)$. Hence $f(xy,\alpha) = f(yx,\alpha)$, completing the proof. □

Definition 4.15. An Ω-fuzzy set f in S is called an *Ω-fuzzy interior ideal* of S if it satisfies

(i) f is an Ω-fuzzy subsemigroup of S,
(ii) $(\forall \alpha \in \Omega)\,(\forall w, x, y \in S)\,(f(xwy,\alpha) \geq f(w,\alpha)\})$.

If f is an Ω-fuzzy bi-ideal (resp. Ω-fuzzy interior ideal) of S, then the fuzzy set Φ in S^Ω given in Theorem 3.9 is a fuzzy bi-ideal (resp. fuzzy interior ideal) of S^Ω.

Theorem 4.16. *Let μ be a nonconstant Ω-fuzzy subsemigroup* (resp. *Ω-fuzzy left (right) ideal, Ω-fuzzy bi-ideal*) *of S and let a be an element of S such that $\mu(a,\alpha) > \mu(x,\alpha)$ for all $x \in S$ and $\alpha \in \Omega$. Then an Ω-fuzzy set ν in S given by $\nu(x,\alpha) = \frac{\mu(x,\alpha)}{\mu(a,\alpha)}$ for all $x \in S$ and $\alpha \in \Omega$ is an Ω-fuzzy subsemigroup* (resp. *Ω-fuzzy left (right) ideal, Ω-fuzzy bi-ideal*) *of S and $\nu(a,\alpha) = 1$.*

Proof. Straightforward. □

Theorem 4.17. *Every Ω-fuzzy ideal is both an Ω-fuzzy bi-ideal and an Ω-fuzzy interior ideal.*

Proof. Let f be an Ω-fuzzy ideal of S. Obviously f is an Ω-fuzzy subsemigroup of S. Let $x, y, z \in S$ and $\alpha \in \Omega$. Then

$$f(xyz, \alpha) = f((xy)z, \alpha) \geq f(z, \alpha),$$
$$f(xyz, \alpha) = f(x(yz), \alpha) \geq f(x, \alpha),$$

and so $f(xyz, \alpha) \geq \min\{f(x, \alpha), f(z, \alpha)\}$. Thus f is an Ω-fuzzy bi-ideal of S. Now let $x, y, w \in S$ and $\alpha \in \Omega$. Then

$$f(xwy, \alpha) \geq f(wy, \alpha) \geq f(w, \alpha),$$

and thus f is an Ω-fuzzy interior ideal of S. □

If S has the identity e, we write it by S^e.

Theorem 4.18. *Every Ω-fuzzy interior ideal of S^e is an Ω-fuzzy ideal of S^e.*

Proof. Let f be an Ω-fuzzy interior ideal of S^e and let $x, y \in S$ and $\alpha \in \Omega$. Then

$$f(xy, \alpha) = f(xye, \alpha) \geq f(y, \alpha),$$
$$f(xy, \alpha) = f(exy, \alpha) \geq f(x, \alpha),$$

which shows that f is an Ω-fuzzy ideal of S^e. □

Let f be an Ω-fuzzy set in S. The smallest Ω-fuzzy left (resp. right) ideal of S containing f is called an *Ω-fuzzy left* (resp. *right*) *ideal generated by* f and is denoted by $\langle f \rangle_l$ (resp. $\langle f \rangle_r$).

Theorem 4.19. *Let f be an Ω-fuzzy set in S^e. Then*

$$\langle f \rangle_l(w, \alpha) = \sup\{f(x_2, \alpha) \mid w = x_1x_2,\ x_1, x_2 \in S^e\}$$

and

$$\langle f \rangle_r(w, \alpha) = \sup\{f(x_1, \alpha) \mid w = x_1x_2,\ x_1, x_2 \in S^e\}$$

for all $w \in S^e$ and $\alpha \in \Omega$.

Proof. Let g be an Ω-fuzzy set in S^e given by

$$g(w, \alpha) = \sup\{f(x_2, \alpha) \mid w = x_1x_2,\ x_1, x_2 \in S^e\}$$

for all $w \in S^e$ and $\alpha \in \Omega$. Then

$$g(w, \alpha) \geq \sup\{f(w, \alpha) \mid w = ew\} = f(w, \alpha).$$

Now we have

$$\begin{aligned} g(xy, \alpha) &= \sup\{f(x_2, \alpha) \mid xy = x_1x_2\} \\ &\geq \sup\{f(z_2, \alpha) \mid xy = (xz_1)z_2,\ y = z_1z_2\} \\ &\geq \sup\{f(z_2, \alpha) \mid y = z_1z_2\} = g(y, \alpha), \end{aligned}$$

which means that g is an Ω-fuzzy left ideal of S^e. Let h be an Ω-fuzzy left ideal of S^e containing f. Then $f(x, \alpha) \leq h(x, \alpha)$ for all $x \in S^e$ and $\alpha \in \Omega$, and hence

$$\begin{aligned} g(w, \alpha) &= \sup\{f(x_2, \alpha) \mid w = x_1x_2\} \leq \sup\{h(x_2, \alpha) \mid w = x_1x_2\} \\ &\leq \sup\{h(x_1x_2, \alpha) \mid w = x_1x_2\} = h(w, \alpha) \end{aligned}$$

for all $w \in S^e$ and $\alpha \in \Omega$. Therefore we have

$$\langle f\rangle_l(w,\alpha) = \sup\{f(x_2,\alpha) \mid w = x_1x_2,\ x_1,x_2 \in S^e\}$$

for all $w \in S^e$ and $\alpha \in \Omega$. Similarly we conclude that

$$\langle f\rangle_r(w,\alpha) = \sup\{f(x_1,\alpha) \mid w = x_1x_2,\ x_1,x_2 \in S^e\}$$

for all $w \in S^e$ and $\alpha \in \Omega$. □

Let f be an Ω-fuzzy set in S. The smallest Ω-fuzzy interior ideal of S containing f is called an *Ω-fuzzy interior ideal generated by* f and is denoted by $\langle f\rangle_I$.

Theorem 4.20. *Let f be an Ω-fuzzy set in S^e. Then*

$$\langle f\rangle_I(w,\alpha) = \sup\{f(x_2,\alpha) \mid w = x_1x_2x_3\}$$

for all $w \in S^e$ and $\alpha \in \Omega$.

Proof. Let g be an Ω-fuzzy set in S^e given by

$$g(w,\alpha) = \sup\{f(x_2,\alpha) \mid w = x_1x_2x_3\}.$$

For any $x,y \in S^e$ and $\alpha \in \Omega$, we have

$$\begin{aligned} g(xy,\alpha) &= \sup\{f(x_2,\alpha) \mid xy = x_1x_2x_3\} \\ &\geq \sup\{f(z_2,\alpha) \mid xy = z_1z_2(z_3y),\ x = z_1z_2z_3\} \\ &= g(x,\alpha). \end{aligned}$$

Similarly we get $g(xy,\alpha) \geq g(y,\alpha)$. Hence

$$g(xy,\alpha) \geq \min\{g(x,\alpha), g(y,\alpha)\},$$

and so g is an Ω-fuzzy subsemigroup of S^e. For any $x \in S^e$ and $\alpha \in \Omega$, we obtain

$$\begin{aligned} g(x,\alpha) &= \sup\{f(x_2,\alpha) \mid x = x_1x_2x_3\} \\ &\geq \sup\{f(x,\alpha) \mid x = exe\} = f(x,\alpha), \end{aligned}$$

that is, g contains f. Let $w,x,y \in S^e$ and $\alpha \in \Omega$. Then

$$\begin{aligned} g(xwy,\alpha) &= \sup\{f(x_2,\alpha) \mid xwy = x_1x_2x_3\} \\ &\geq \sup\{f(z_2,\alpha) \mid xwy = (xz_1)z_2(z_3y),\ w = z_1z_2z_3\} \\ &= g(w,\alpha). \end{aligned}$$

Thus g is an Ω-fuzzy interior ideal of S^e containing f. Let h be an Ω-fuzzy interior ideal of S^e containing f. Then

$$\begin{aligned} g(w,\alpha) &= \sup\{f(x_2,\alpha) \mid w = x_1x_2x_3\} \\ &\leq \sup\{h(x_2,\alpha) \mid w = x_1x_2x_3\} \\ &\leq \sup\{h(x_1x_2x_3,\alpha) \mid w = x_1x_2x_3\} \\ &= h(w,\alpha). \end{aligned}$$

This completes the proof. □

Let f be an Ω-fuzzy set in S. The smallest Ω-fuzzy bi-ideal of S containing f is called an *Ω-fuzzy bi-ideal generated by* f and is denoted by $\langle f\rangle_B$.

Theorem 4.21. *Let f be an Ω-fuzzy set in S^e such that $f(e,\alpha) \geq f(x,\alpha)$ for all $x \in S$ and $\alpha \in \Omega$. Then*

$$\langle f\rangle_B(w,\alpha) = \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\}$$

for all $w \in S^e$ and $\alpha \in \Omega$.

Proof. Let g be an Ω-fuzzy set in S^e given by

$$g(w,\alpha) = \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\}.$$

For any $w \in S^e$ and $\alpha \in \Omega$,

$$\begin{aligned} g(w,\alpha) &= \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\} \\ &\geq \sup\{\min\{f(e,\alpha), f(w,\alpha)\} \mid w = eew\} \\ &= \sup\{f(w,\alpha) \mid w = eew\} = f(w,\alpha), \end{aligned}$$

and so g contains f. Let $x, y, z \in S^e$ and $\alpha \in \Omega$. Then

$$g(x,\alpha) = \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid x = x_1x_2x_3\},$$

$$g(z,\alpha) = \sup\{\min\{f(z_1,\alpha), f(z_3,\alpha)\} \mid z = z_1z_2z_3\},$$

and hence

$$\begin{aligned} &\min\{g(x,\alpha), g(z,\alpha)\} \\ &\quad = \sup\{\min\{\min\{f(x_1,\alpha), f(x_3,\alpha)\}, \min\{f(z_1,\alpha), f(z_3,\alpha)\}\} \mid \\ &\qquad\qquad x = x_1x_2x_3,\ z = z_1z_2z_3\} \\ &\quad = \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha), f(z_1,\alpha), f(z_3,\alpha)\} \mid \\ &\qquad\qquad x = x_1x_2x_3,\ z = z_1z_2z_3\} \\ &\quad \leq \sup\{\min\{f(x_1,\alpha), f(z_3,\alpha)\} \mid xyz = x_1(x_2x_3yz_1z_2)z_3, \\ &\qquad\qquad x = x_1x_2x_3, z = z_1z_2z_3\} \\ &\quad \leq \sup\{\min\{f(u_1,\alpha), f(u_3,\alpha)\} \mid xyz = u_1u_2u_3\} = g(xyz,\alpha). \end{aligned}$$

If we take $y = e$, then $g(xz,\alpha) \geq \min\{g(x,\alpha), g(z,\alpha)\}$. Hence g is an Ω-fuzzy bi-ideal of S^e. Let h be an Ω-fuzzy bi-ideal of S^e containing f. Then

$$\begin{aligned} g(w,\alpha) &= \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\} \\ &\leq \sup\{\min\{h(x_1,\alpha), h(x_3,\alpha)\} \mid w = x_1x_2x_3\} \\ &\leq \sup\{\min\{h(x_1x_2x_3,\alpha)\} \mid w = x_1x_2x_3\} \\ &= h(w,\alpha) \end{aligned}$$

for all $w \in S^e$ and $\alpha \in \Omega$. This completes the proof. □

Let f be an Ω-fuzzy set in S^e and assume that S^e is regular in Theorem 4.21. Then

$$\begin{aligned} g(w,\alpha) &= \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\} \\ &\geq \sup\{\min\{f(w,\alpha), f(w,\alpha)\} \mid w = wxw\} \\ &= f(w,\alpha). \end{aligned}$$

Hence we have the following theorem.

Theorem 4.22. *Let f be an Ω-fuzzy set in S^e. If S^e is regular, then*

$$\langle f\rangle_B(w,\alpha) = \sup\{\min\{f(x_1,\alpha), f(x_3,\alpha)\} \mid w = x_1x_2x_3\}$$

for all $w \in S^e$ and $\alpha \in \Omega$.

5. Conclusions

In this paper we have defined the notions of Ω-fuzzy subsemigroups, Ω-fuzzy left (right) ideals, Ω-fuzzy bi-ideals and Ω-fuzzy interior ideals in semigroups by by using a set Ω. We have described an Ω-fuzzy subsemigroup by using a fuzzy subsemigroup and vice versa. We have stated how the homomorphic images and inverse images of Ω-fuzzy subsemigroups become Ω-fuzzy subsemigroups. We have dealt with the notion of an Ω-fuzzy left (right) ideal (an Ω-fuzzy bi-ideal, an Ω-fuzzy interior ideal) generated by an Ω-fuzzy set in semigroups, and examined the depictions of them. Our future work will focus on studying the intuitionistic Ω-fuzzy structure of several ideals in semigroups.

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